A NOTE ON THE LICHNEROWICZ VANISHING THEOREM FOR PROPER ACTIONS

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Abstract. We prove a Lichnerowicz type vanishing theorem for non-compact spin manifolds admiting proper cocompact actions. This extends a previous result of Ziran Liu who proves it for the case where the acting group is unimodular.

0. Introduction

A classical theorem of Lichnerowicz [3] states that if an even dimensional closed smooth spin manifold admits a Riemannian metric of positive scalar curvature, then the index of the associated Dirac operator vanishes. In this note we prove an extension of this vanishing theorem to the case where a (possibly non-compact) spin manifold M admiting a proper cocompact action by a locally compact group G.

To be more precise, recall that for such an action, a so called G-invariant index has been defined by Mathai-Zhang in [5]. Thus it is natural to ask whether this index vanishes if M carries a G-invariant Riemannian metric of positive scalar curvature. Such a result has indeed been proved by Liu in [4] for the case of unimodular G. In this note we extend Liu's result to the case of general G.

We will recall the defnition of the Mathai-Zhang index [5] and state the main result as Theorem 1.2 in Section 1; and then prove Theorem 1.2 in Section 2.

1. A vanishing theorem for the Mathai-Zhang index

Let M be an even dimensional spin manifold. Let G be a locally compact group which acts on M properly and cocompactly, where by proper we mean that the map

$$G \times M \to M \times M, \quad (g, x) \mapsto (x, gx),$$

is proper (the pre-image of a compact subset is compact), while by cocompact we mean that the quotient M/G is compact. We also assume that G preserves the spin structure on M.

Given a G-invariant Riemannian metric g^{TM} (cf. [5, (2.3)]), we can construct canonically a G-equivariant Dirac operator $D : \Gamma(S(TM)) \to \Gamma(S(TM))$ (cf. [2] and [5]), acting on the Hermitian spinor bundle $S(TM) = S_+(TM) \oplus S_-(TM)$. Let $D_\pm : \Gamma(S_\pm(TM)) \to \Gamma(S_\mp(TM))$ be the obvious restrictions.

Let $\|\cdot\|_0$ be the standard L^2-norm on $\Gamma(S(TM))$, let $\|\cdot\|_1$ be a (fixed) G-invariant Sobolev 1-norm. Let $\mathbf{H}^0(M, S(TM))$ be the completion of $\Gamma(S(TM))$ under$\|\cdot\|_0$. Let $\Gamma(S(TM))^G$ denote the space of G-invariant smooth sections of $S(TM)$.

Recall that by the compactness of M/G, there exists a compact subset Y of M such that $G(Y) = M$ (cf. [6, Lemma 2.3]). Let U, U' be two open subsets of M such that $Y \subset$

U and that the closures $\overline{U}$ and $\overline{U'}$ are both compact in M, and that $\overline{U} \subset U'$. Following [5], let $f \in C^\infty(M)$ be a nonnegative function such that $f|_U = 1$ and $\mathrm{Supp}(f) \subset U'$.

Let $\mathbf{H}^0_f(M, S(TM))^G$ and $\mathbf{H}^1_f(M, S(TM))^G$ be the completions of $\{fs : s \in \Gamma(S(TM))^G\}$ under $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively. Let P_f denote the orthogonal projection from $\mathbf{H}^0(M, S(TM))$ to $\mathbf{H}^0_f(M, S(TM))^G$. Clearly, $P_f D$ maps $\mathbf{H}^1_f(M, S(TM))^G$ into $\mathbf{H}^0_f(M, S(TM))^G$.

We recall a basic result from [5, Proposition 2.1].

Proposition 1.1. *The operator $P_f D : \mathbf{H}^1_f(M, S(TM))^G \to \mathbf{H}^0_f(M, S(TM))^G$ is a Fredholm operator.*

It has been shown in [5] that $\mathrm{ind}(P_f D_+)$ is independent of the choice of the cut-off function f, as well as the G-invariant metric involved. Following [5, Definition 2.4], we denote $\mathrm{ind}(P_f D_+)$ by $\mathrm{ind}_G(D_+)$.

The main result of this note can be stated as follows.

Theorem 1.2. *If there is a G-invariant metric g^{TM} on TM such that its scalar curvature k^{TM} is positive over M, then $\mathrm{ind}_G(D_+) = 0$.*

Remark 1.3. If G is unimodular, then Theorem 1.2 has been proved in [4]. Our proof of Theorem 1.2 combines the method in [4] with a simple observation that in order to prove the vanishing of the index, one need not restrict to self-adjoint operators.

2. Proof of Theorem 1.2

Following [5, (2.16)], let $\widetilde{D}_{f,\pm} : \mathbf{H}^1_f(M, S_\pm(TM))^G \to \mathbf{H}^0_f(M, S_\mp(TM))^G$ be defined by that for any $s \in \Gamma(S_\pm(TM))^G$,

$$\widetilde{D}_{f,\pm}(fs) = f\, D_\pm s. \tag{2.1}$$

Since one verifies easily that (cf. [5, (4.2)])

$$\widetilde{D}_{f,\pm}(fs) - P_f D_\pm(fs) = -P_f\left(c(df)s\right), \tag{2.2}$$

one sees that $\widetilde{D}_{f,\pm}$ is a compact perturbation of $P_f D_\pm$. Thus, one has

$$\mathrm{ind}\left(\widetilde{D}_{f,+}\right) = \mathrm{ind}\left(P_f D_+\right). \tag{2.3}$$

Now by (2.1), if $fs \in \ker(\widetilde{D}_{f,+})$, then $s \in \ker(D_+)$. Thus, by the standard Lichnerowicz formula [3], one has (cf. [1, pp. 112] and [4, (3.6)])

$$\frac{1}{2}\Delta\left(|s|^2\right) = \left|\nabla^{S_+(TM)} s\right|^2 + \frac{k^{TM}}{4}|s|^2 \geq \frac{k^{TM}}{4}|s|^2, \tag{2.4}$$

where Δ is the negative Laplace operator on M and $\nabla^{S_+(TM)}$ is the canonical Hermitian connection on $S_+(TM)$ induced by g^{TM}.

As has been observed in [4], since the G-action on M is cocompact and $|s|$ is clearly G-invariant, there exists $x \in M$ such that

$$|s(x)| = \max\{|s(y)| : y \in M\}. \tag{2.5}$$

By the standard maximum principle, one has at x that

$$\Delta\left(|s|^2\right) \leq 0. \tag{2.6}$$

Combining (2.6) with (2.4), one sees that if $k^{TM} > 0$ over M, one has

$$s(x) = 0, \tag{2.7}$$

which implies that $s \equiv 0$ on M. Thus, one has $\ker(\widetilde{D}_{f,+}) = \{0\}$, and, consequently,

$$\operatorname{ind}\left(\widetilde{D}_{f,+}\right) \leq 0. \tag{2.8}$$

On the other hand, for any $s,\ s' \in \Gamma(S(TM))$, one verifies that

$$\langle fDs, fs' \rangle = \left\langle s, D\left(f^2 s'\right)\right\rangle = \langle fs, D\left(fs'\right) + c(df)s' \rangle . \tag{2.9}$$

Let $\widehat{D}_{f,\pm} : \mathbf{H}^1_f(M, S_\pm(TM))^G \to \mathbf{H}^0_f(M, S_\mp(TM))^G$ be defined by that for any $s \in \Gamma(S_\pm(TM))^G$,

$$\widehat{D}_{f,\pm}(fs) = P_f\left(D_\pm(fs) + c(df)s\right). \tag{2.10}$$

Clearly, $\widehat{D}_{f,+}$ is a compact perturbation of $P_f D_+$. Thus one has

$$\operatorname{ind}\left(\widehat{D}_{f,+}\right) = \operatorname{ind}\left(P_f D_+\right). \tag{2.11}$$

Now by (2.9), one sees that the formal adjoint of $\widehat{D}_{f,+}$ is $\widetilde{D}_{f,-}$, while by proceeding as in (2.4)-(2.7), one finds that $\ker(\widetilde{D}_{f,-}) = \{0\}$. Thus, one has

$$\operatorname{ind}\left(\widehat{D}_{f,+}\right) \geq 0. \tag{2.12}$$

From (2.3), (2.8), (2.11) and (2.12), one gets $\operatorname{ind}\left(P_f D_+\right) = 0$, which completes the proof of Theorem 1.2.

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